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AN APPLICATION OF ELLIPTIC FUNCTIONS TO PEAUCELLIER'S LINK-WORK (INVERSOR).

BY ARNOLD EMCH.

IN a recent article which appeared in the ANNALS OF MATHEMATICS,[*] I have studied a certain link-work and its relation with the elliptic integral of the first kind. By a geometrical transformation a class of circular series was

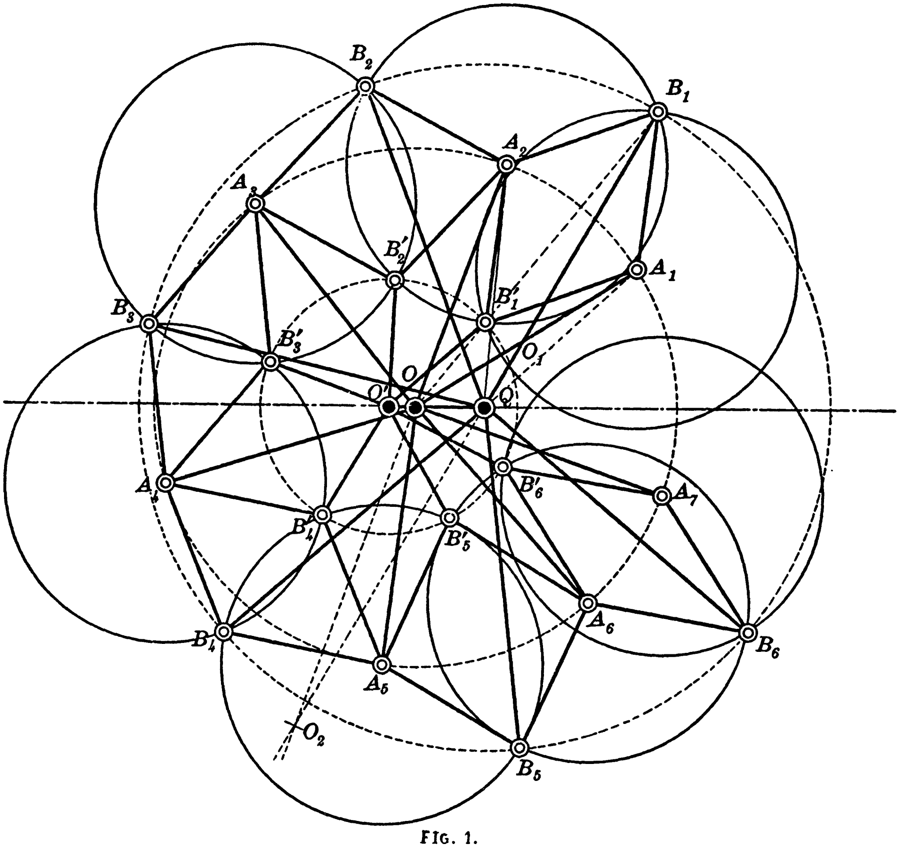

FIG. 1.

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[*] "Illustration of the Elliptic Integral of the First Kind by a Certain Link-Work," ser. 2, vol. 1 (1900), p. 81.

obtained which led to *Poncelet's poristic polygons*, and suggested their equivalence with *Steiner's circular series*. By a slight generalization the link-work may be given such a form that, implicitly, it also includes Steiner's series. In this general form the element of the link-work is represented by *Peaucellier's Cell or Inversor.*[*]

1. The Link-Work. Assume in Fig. 1, the links $OA_1 = OA_2 = r_1$, $A_1B_1 = B_1A_2 = A_2B_1' = B_1'A_1 = r_2$, $QB_1 = R$, and $OQ = e$. The points O and Q are fixed, while all the others are movable. The arrangement of this link-work is evidently that of *Peaucellier's Inversor*. During the motion B_1 describes a circle having Q as a centre and QB_1 as a radius. Now, according to the properties of Peaucellier's Inversor $OB_1 \cdot OB_1' = r_1^2 - r_2^2 =$ constant; consequently the point B_1' also describes a circle, which is inverse to the circle described by B_1 in an inversion having O as a centre and $\sqrt{r_1^2 - r_2^2}$ as a radius. Designating the variable distances QA_1 and QA_2 by ρ_1 and ρ_2, respectively, and putting $\rho_1^2 = x$, $\rho_2^2 = y$, $R + r_2 = a$, $r_2 + e = b$, $r_2 - e = c$, $R - r_2 = d$; furthermore designating by u and v the elliptic integrals

$$u = \int^x \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}}, \tag{1}$$

$$v = \int^y \frac{dy}{\sqrt{(y-a)(y-b)(y-c)(y-d)}}, \tag{2}$$

the fundamental relation

$$v - u = h \text{ (constant)} \tag{3}$$

may be found. The proof for this relation is essentially the same as the proof of the corresponding relation in my previous paper.[†] By inversion of the integrals (1) and (2) the elliptic functions $x = \lambda(u)$, $y = \lambda(v)$ are obtained. Thus the variables ρ_1 and ρ_2 may be expressed by elliptic functions, whose arguments have a constant difference. Here as there, other cells of equal size $(OA_2B_2A_3B_2' \cdot QB_2)$, $(OA_3B_3A_4B_3' \cdot QB_3)$, $\cdots\cdot$ may be added to the first, so as to form a general link-work which may close or may not close. Similar analysis to that employed in the former case leads to the corresponding result:

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[*] A description of this particular link-work may be found in every modern text-book on kinematics. The constructions that it performs were proposed by Peaucellier in the *Nouvelles Annales de Mathématiques*, ser. 2, vol. 3 (1864), p. 414, and the link-work was published by him in the same journal, ser. 2, vol. 12 (1873), p. 71.

[†] *l. c.*, p. 84.

If a link-work of the prescribed kind, based upon two fixed circles (centres O and Q, radii OA_1 *and* QB_1*) closes and contains n cells, then every other link-work based upon the same two circles closes and contains n cells.*

It is clear that the fundamental relation (3) also holds in the cases of limited and unlimited link-works. The previous theorem, however, only holds for a closed link-work. Without entering into a further discussion of the various special cases which may present themselves, and which were treated in the article referred to above, I shall now explain a geometrical transformation of the link-work which in its generality contains Steiner's and Poncelet's configurations.

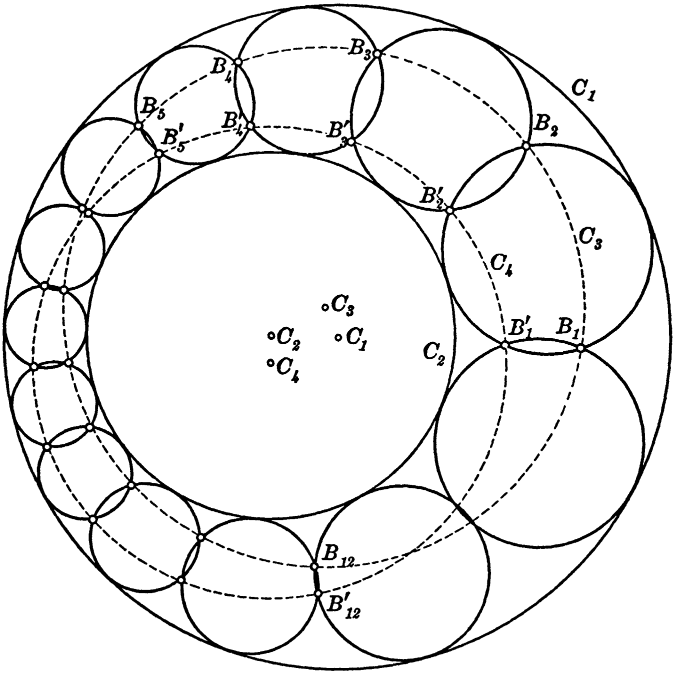

Fig. 2.

2. Geometrical Transformation of the Link-Work. In Fig. 1, with A_1, A_2, A_3, . . . as centres and r_2 as a radius describe a series of circles. From the figure it is seen that the two circles having A_i and A_{i+1} as centres pass through the points B_i and B_i'. The properties of closing of these series are evidently the same as those of the link-work. All circles of the series are tangent to two concentric circles. The figure can be generalized by an inversion and we have immediately the result:

If each pair of consecutive circles of a series of circles, which all touch two fixed circles C_1 *and* C_2, *intersect in two points* B_i *and* B'_i, *and if the points* B_1, B_2, B_3, . . . *are situated on a circle* C_3, *then the points* B'_1, B'_2, B'_3, . . . *are also situated on a circle* C_4* (Fig. 2).

As the geometrical transformation of the link-work does not affect the properties of closing of the above series of circles we may state, in analogy with the link-work:

If a series of circles (generalized Steinerian series) of the prescribed kind, based upon two fixed circles C_1 *and* C_2 *and a third circle* C_3 *(or* C_4*), closes and contains n circles, then every other series of circles based in the same manner upon the same three circles closes and contains n circles.*

3. Steiner's Series and Poncelet's Polygons. By special disposition and by assuming some of the given circles as points or straight lines a great variety of circular series may be obtained. If the circles C_1, C_2 and C_3 are parts of a pencil of circles, then C_4 belongs to the same pencil. The series of circles obtained by this arrangement have been considered by Steiner.† One of the most interesting cases arises if one of the circles C_1 and C_2, for instance C_2, degenerates into point O. All circles of the series pass through O, and the circle C_4 coincides with O. Any inversion having O as a centre transforms all circles of the series into straight lines which are inscribed in a circle C'_3 and circumscribed about a circle C'_1, *i. e.* the limited portions of these lines form a polygon which is inscribed in one and circumscribed about another circle. The properties of closing of these polygons, which are called Poncelet's polygons, are the same as those of the general series of circles. Poncelet's constructions also result directly from a geometrical transformation of the link-work in which $r_1 = r_2$. This case was exclusively studied in the article to which I have referred in the beginning.

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* This statement may be generalized in such a manner that instead of a circle C_3 any curve is assumed. From Fig. 1 it can easily be proved that in this case B'_1, B'_2, B'_3, . . . are situated on a curve which is the inverse of the first with regard to the centre O. The special case above has been formulated in view of its subsequent application.

† *Werke*, vol. 1, pp. 19–76.